Analytical Expression for the Gradient b-Factor in NMR Diffusion Imaging.

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Introduction:
Measurements of molecular diffusion using MRI has generated great interest, especially in functional assessment, tissue characterization, and treatment monitoring. The estimation of the diffusion tensor, D, in anisotropic media by NMR spectroscopy and imaging requires the determination of a b-matrix from the gradient pulse sequence:

\[
\ln \left( \frac{S(b)}{S(0)} \right) = -\sum_{i=1}^{2} \sum_{j=1}^{2} b_{ij} D_{ij} \]  

where \( b_{ij} \) are components of the b-matrix, \( D_{ij} \) are components of the effective self-diffusion tensor, \( D \) and \( S(b) \) and \( S(0) \) are echo intensities with and without applied diffusion gradients, respectively.

We present analytical expressions of the b-matrix for a 2D-FT spin-echo imaging sequence, and show the effect of ignoring cross-terms in estimating the apparent diffusion tensor.

Expression for the b-factors:
Using definitions for the gradient pulse sequences\(^{3,4} \):

\[
G(t) = \left( G_x(t), G_y(t), G_z(t) \right), \quad F(t) = \left( F(t)^{c+} \right)^{\dagger}, \quad \text{and } f = F(1/2 \text{TE}) \]

we relate the echo intensity to the diffusion tensor as:

\[
\ln \left( \frac{S(b)}{S(0)} \right) = -\gamma \int \left( F(t) - 2 S(0) \right) \cdot D \cdot \left( F(t) - 2 S(0) \right)^{\dagger} \text{dt},
\]

where TE is the echo time, \( \gamma \) is the gyromagnetic ratio, \( \xi < 1/2 \) \( \text{TE} = 0 \), \( \xi > 1/2 \) \( \text{TE} = 1 \). The b-matrix elements are derived from the expression:

\[
b = -\gamma \int \left( F(t) - 2 S(0) \right) \cdot \left( F(t) - 2 S(0) \right)^{\dagger} \text{dt}.
\]

A symbolic manipulation program similar to that employed by Price et al.\(^{5} \), was used to derive analytical expression for the b-matrix. A generalized 2D-FT spin-echo imaging sequence, shown in Figure 1, was synthesized from a library of sinusoidal and trapezoidal gradient pulses, which includes all gradient pulses that are typically encountered. The imaging gradient intensities are: Gsl, 60° slice-selection gradient; Gpe, Gdp, or Gsf, phase-encoding, read-dephasing or slice-refocusing gradients, respectively; Gsd, Gdp, or Gsd, diffusion gradients in the r = read, p = phase, and s = slice directions, respectively; Gcs, Gcp, or Gcs, crusher gradients in the read, phase, and slice directions, respectively; and G60, 180° slice-selection gradient; and Gro, readout gradient. The timing parameters, \( t_{ij} \), are listed in Table 1. The b-matrix is symmetric (\( b_{ij} = b_{ji} \)) so only six independent b-matrix elements are derived:

\[
\begin{align*}
 b_{11} &= \gamma \int \left( G_x^2 G_{sd} G_{dp} + 2 G_{sd} G_{dp} G_{sd} + 2 G_{sd} G_{dp} G_{dp} + G_{sd}^2 T_33 + G_{dp}^2 T_33 + 2 G_{sd} G_{dp} T_33 + 2 G_{dp} G_{dp} T_33 \right) dt, \\
 b_{12} &= \gamma \int \left( G_x G_y G_{sd} + G_x G_y G_{dp} + 2 G_{sd} G_{dp} G_{sd} + 2 G_{sd} G_{dp} G_{dp} + G_{sd}^2 T_33 + G_{dp}^2 T_33 + 2 G_{sd} G_{dp} T_33 + 2 G_{dp} G_{dp} T_33 \right) dt, \\
 b_{13} &= \gamma \int \left( G_x G_z G_{sd} + G_x G_z G_{dp} + 2 G_{sd} G_{dp} G_{sd} + 2 G_{sd} G_{dp} G_{dp} + G_{sd}^2 T_33 + G_{dp}^2 T_33 + 2 G_{sd} G_{dp} T_33 + 2 G_{dp} G_{dp} T_33 \right) dt, \\
 b_{23} &= \gamma \int \left( G_y G_z G_{sd} + G_y G_z G_{dp} + 2 G_{sd} G_{dp} G_{sd} + 2 G_{sd} G_{dp} G_{dp} + G_{sd}^2 T_33 + G_{dp}^2 T_33 + 2 G_{sd} G_{dp} T_33 + 2 G_{dp} G_{dp} T_33 \right) dt, \\
 b_{33} &= \gamma \int \left( G_z^2 G_{sd} G_{dp} + 2 G_{sd} G_{dp} G_{sd} + 2 G_{sd} G_{dp} G_{dp} + G_{sd}^2 T_33 + G_{dp}^2 T_33 + 2 G_{sd} G_{dp} T_33 + 2 G_{dp} G_{dp} T_33 \right) dt.
\end{align*}
\]

Discussion and Concluding Remarks:
Ignoing cross-terms in calculating the b-matrix introduces significant errors into the estimate of the apparent diffusion tensor. Additionally, in anisotropic media, the off-diagonal components of the b-matrix are required to estimate the components of the diffusion tensor, as described by Basser et al., and should not be ignored whether or not there are specific diffusion gradients. These analytical expressions for the b-matrix permit the accurate determination of all the elements of the diffusion tensor. The linear fit of the imaging data demonstrate that the b-matrix components were calculated correctly in each of the three coordinate directions.

References:

Graph 1. Generalized pulse sequence used to generate b-matrix.

\[\begin{align*}
\Delta_t &= \frac{\gamma^2}{2} \left( G_{x_1} \Delta_2 T_{12} + G_{x_2} \Delta_3 T_{13} + G_{y_1} \Delta_2 T_{12} + G_{y_2} \Delta_3 T_{13} + G_{z_1} \Delta_2 T_{12} + G_{z_2} \Delta_3 T_{13} \right) \\
\Delta_s &= \frac{\gamma^2}{2} \left( G_{x_1} \Delta_2 T_{12} + G_{x_2} \Delta_3 T_{13} + G_{y_1} \Delta_2 T_{12} + G_{y_2} \Delta_3 T_{13} + G_{z_1} \Delta_2 T_{12} + G_{z_2} \Delta_3 T_{13} \right) \\
\end{align*}\]

Table 1. Time parameters used in calculating the b-matrix. \( t_i \) is time gradient pulse turns on during pulse sequence.